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Fredholm and Volterra Integral Equations

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ABSTRACT

We will focus on Fredholm and Volterra integral equations. We have examined the development of integral equations that has significant applicability in physical problems. A multitude of initial and boundary value issues can be converted into integral equations. Mathematical physics problems are typically regulated by integral equations. There are several categories of integral equations. Singular integral equations are highly beneficial in numerous physical issues, including elasticity, fluid mechanics, and electromagnetic theory.

Keywords: Fredholm integral equations, Volterra integral equations, Fredholm integral-differential equation, Integro-differential equations.

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Fredholm integral equations:

In this Section, we shall be concerned with the Fredholm integral equations.

$$\mathbf{u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \qquad a \le x \le b} \qquad (3.1)$$

$$\mathbf{u(x) = f(x) + \lambda \int_a^b K(x,t)\{u(t)\}^2dt \qquad a \le x \le b} \qquad (3.2)$$

$$\mathbf{f(x) = \int_a^b K(x,t)u(t)dt \qquad a \le x \le b} \qquad (3.3)$$

All are Fredholm integral equations. Equation (3.1) represents the nonhomogeneous Fredholm linear integral equation of the second kind; equation (3.2) denotes the Fredholm linear integral equation of the first kind; and equation (3.3) signifies the Fredholm nonlinear integral equation of the second kind. In all these instances, K(x,t) and f(x) are defined functions. K(x,t) denotes the kernel of the integral equation formed within the rectangle R, where $a \le x \le b$ and $a \le t \le b$, and f(x) represents the forcing term defined for $a \le x \le b$.If f(x)=0, then the equations are classified as homogeneous. The functions u, f, and K may be complex-valued. Linear and nonlinear integral equations are characterised by the presence of the unknown function in a linear or nonlinear manner beneath the integral sign. The parameter λ is a predetermined value. In the subsequent section, we will examine the several approaches of solutions for 1.1. The technique of iterative approximations: Neumann series

$$\mathbf{u(x) = f(x) + \lambda \int_a^b k(x,t)\, u(t)\, dt}$$

$$\mathbf{u_1(x) = f(x) + \lambda \int_a^b k(x,t)\, u_0(t)dt}$$

$$\mathbf{u_2(x) = f(x) + \lambda \int_a^b k(x,t)\, u_1(t)\, dt}$$

$$\mathbf{u_n(x) = f(x) + \lambda \int_a^b K(x,t)u_{n-1}\ dt \qquad n \ge 1}$$

$$\mathbf{u(x) = \lim_{n\to\infty} u_n(x)}$$

Example 1:
Solve the Fredholm integral equation

$$\mathbf{u(x) = 1 + \int_0^1 xu(t)\, dt}$$

by using successive approximation method.

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Solution

Let us consider the zeroth approximation is $\mathbf{u_0}$ (x) = 1, and then the first approximation can be computed as

$$\mathbf{u_1(x) = 1 + \int_0^1 xu_0(t)\,dt}$$

$$\mathbf{= 1 + \int_0^1 x\,dt}$$

$$\mathbf{= 1 + x}$$

Proceeding in this manner, we find

$$\mathbf{u_2(x) = 1 + \int_0^1 xu_1(t)\,dt}$$

$$\mathbf{= 1 + \int_0^1 x\,(1+t)\,dt}$$

$$\mathbf{= 1 + x\left(1 + \frac{1}{2}\right)}$$

Similarly, the third approximation is

$$\mathbf{u_3(x) = 1 + x\int_0^1 \left(1 + \frac{t}{2}\right)dt}$$

$$\mathbf{= 1 + x\left(1 + \frac{1}{2} + \frac{1}{4}\right)}$$

Thus, we get

$$\mathbf{u_n(x) = 1 + x\left\{1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}\right\}}$$

and hence

$$\mathbf{u(x) = \lim_{n\to\infty} u_n(x)}$$

$$\mathbf{= 1 + \lim_{n\to\infty} x\sum_{k=0}^{n}\frac{1}{2^k}}$$

$$\mathbf{1 + x\left(1 - \frac{1}{2}\right)^{-1}}$$

$$\mathbf{= 1 + 2x}$$

Example 2:

use the successive approximation to solve Fredholm integral

u(x) = sin x + $\int_0^{\frac{\pi}{2}} \mathbf{\sin x \cos t\, u(t) dt}$

Solution:

$$\mathbf{u_0(x) = 1}$$

$\mathbf{u_1(x) = \sin x + \sin x \int_0^{\frac{\pi}{2}} \cos t\, dt}$ = 2sin x

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$\mathbf{u_2(x) = sinx + sinx \int_0^{\frac{\pi}{2}} 2\, sin\, t\, cost\, dt}$ = 2sin x

$\mathbf{u_3(x) = 2\, sin\, x}$, $\mathbf{u_4(x) = 2\, sin\, x}$

$\mathbf{u(x)=lim_{n\to\infty}\, u_n\,(x)}$ = 2 sin x

The method of successive substitutions:

u (x)=f(x)+λ$\int_a^b \mathbf{k(x,t)u(t)dt}$

u(x)=f(x)+λ$\int_a^b \mathbf{k(x,t)f(t)dt + \lambda^2 \int_a^b k(x,t) \int_a^b k(t,t_1)}u\mathbf{(t_1)dt_1dt}$

u(x)=f(x)+λ$\int_a^b \mathbf{k(x,t)f(t)dt + \lambda^2 \int_a^b k(x,t) \int_a^b k(t,t_1)u(t_1)dt_1dt +}$
$\mathbf{\lambda^3 \int_a^b k(x,t) \int_a^b k(t,t_1) \int_a^b u(t_1,t_2)u(t_2)dt_2dt_1dt}$

The A domain decomposition technique:
In the decomposition method, we typically articulate the solution of the linear integral problem.

$$\text{u(x)=f(x)} + \lambda \int_a^b \mathbf{k(x,t)u(t)dt}$$

in a series form $\mathbf{u(x) = \sum_{n=0}^{\infty} u_n(x)}$

substituting the decomposition equation in the integral equation

$$\mathbf{\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_a^b k(x,t) \left\{\sum_{n=0}^{\infty} u_n(t)\right\} dt}$$

The components $\mathbf{u_1(x), u_2(x), u_3(x), \dots}$ **of unknown function** $\mathbf{u(x)}$ are completely determined in a recurrence manner if we set

$\mathbf{u_0(x) = f(x)}$

$\mathbf{u_1(x) = \lambda \int_a^b k(x,t)u_0(t)dt}$

$\mathbf{u_2(x) = \lambda \int_a^b k(x,t)u_1(t)dt}$

$\mathbf{u_3(x) = \lambda \int_a^b k(x,t)u_2(t)dt}$

………..=……………..

$\mathbf{u_n(x) = \lambda \int_a^b k(x,t)u_{n-1}(t)dt}$

Fredholm integral-differential equation:
This section will address the dependable techniques employed to resolve Fredholm integral-differential equations. We note that our focus will be on equations

involving separable kernels, where the kernel K(x,t) can be represented as a finite sum of the form

$$\mathbf{K(x,t) = \sum_{k=1}^{n} g_k(x)h_k(t)}$$

It pertains to integro-differential equations in which both differential and integral operators coexist within the same equation. This category of equations was first introduced by Volterra in the early 1900s. Volterra examined population growth, concentrating on hereditary factors, which led to the establishment of integro-differential equations through his research.

$u'(x) = f(x) - \int_0^x (x-t)u(t)dt,\ u(0) = 0$

$u''(x) = g(x) + \int_0^x (x-t)u(t)dt,\ u(0) = 0,\ u' = -1$

$u'(x) = e^{-x} + \int_0^1 xtu(t)dt,\ u(0) = 0$

$u''(x) = h(x) - \int_0^x t\,u'(t)\,dt,\ u(0) = 0,\ u' = 1$

Volterra integro-differential equations:

This section will introduce advanced mathematical techniques for solving Volterra integro-differential equations. We will concentrate on examining the integral equation that features a separable kernel of the type

$\mathbf{K(x,t)} = \sum_{k=1}^{n} \mathbf{g_k}(x)h_k(t)$

The method of series solutions:

We shall examine a standard form of the nth order Volterra integro-differential equation as presented below.

$U^{(n)}(x) = f(x) + g(x)\int_0^x \mathbf{h(t)u(t)dt}\ ,\mathbf{u^{(n)} = b_k}\ ,\mathbf{0 \le k \le (n-1)}$

$u(x) = \sum_{k=0}^{\infty} \mathbf{a_k x^k}$

$\mathbf{(\sum_{k=0}^{\infty} a_k x^k)^{(n)} = f(x) + g(x) \int_0^x (\sum_{k=0}^{\infty} a_k t^k)dt}$

Example 1

Solve the following Volterra integro-differential equation by using the series solution method

$u''(x) = x\cosh x - \int_0^x \mathbf{t}\,u(t)dt,\ u(0) = 0,\ u'(0) = 1$

Solution

$u(x) = \sum_{n=0}^{\infty} \mathbf{a_n x^n}$

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$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = x(\sum_{k=0}^{\infty}\left(\frac{x^{2k}}{2k!}\right) - \int_0^x t(\sum_{n=0}^{\infty} a_n t^n)dt$$

$$2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \cdots$$
$$= x\left(1 + \frac{x^2}{2!} + \frac{x^4}{3!} + \cdots\right) - (\frac{x^3}{3} + \frac{1}{4}\, a_2x^4 + \cdots)$$

$$u(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \text{sinhx}$$

Existence and Uniqueness Theorems:
These theorems delineate the conditions necessary for the existence and uniqueness of solutions to Volterra integro-differential equations. They depend on characteristics such as continuity, differentiability, and boundedness of the functions involved.

Theorem 1: Picard's Iteration Method:
An iterative method in which an estimated solution is progressively improved at each iteration.
Begins with an initial estimate and produces a series of approximations that converge to the precise result.
Convergence is guaranteed if the integral kernel and input function satisfy the Lipschitz continuity condition.
• Banach Fixed-Point Theorem
A contraction mapping in a complete metric space possesses a unique fixed point.
In Volterra equations, a contractive integral operator guarantees the existence of a unique solution.
• Commonly employed to illustrate well-posedness in both linear and nonlinear scenarios.
Schauder Fixed-Point Theorem:
Schauder’s theorem, in contrast to Banach’s theorem, pertains to compact and continuous operators.
• Beneficial in instances where the contraction requirement is unmet.
• Affirms the presence of solutions, however does not invariably assure uniqueness.

Theorem 2: Stability and Asymptotic Behavior:
Stability analysis assesses a system's response to minor perturbations over time and evaluates whether the solution remains constrained.

Lyapunov Stability Theory:
• Entails the formulation of a Lyapunov function, an energy-like metric utilized to evaluate stability.
• If the Lyapunov function is bounded, the system is stable.

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• Commonly utilized in control theory and dynamical systems exhibiting memory effects.

Grönwall's Inequality:
• A crucial instrument for assessing upper limits on solutions.
• Facilitates the demonstration of uniqueness and stability through the regulation of the integral component.
• Commonly employed to ascertain the well-posedness of integro-differential equations.

Mittag-Leffler Stability:
A generalization of exponential stability, frequently utilized in fractional Volterra equations.
• Employs the Mittag-Leffler function in lieu of an exponential function for stability analysis.
• Implemented in systems where memory effects result in power-law decay rather than exponential decay.

Theorem 3: Perturbation Techniques:
These methods examine the impact of minor alterations in system parameters on the behavior of solutions.

Standard Perturbation Theory:
• Assumes that the answer can be articulated as a power series in relation to a diminutive parameter.
• Functions effectively for issues when minor fluctuations in parameters do not significantly impact the outcome.
• Frequently utilized in physics and engineering challenges with negligible external impacts.
Singular Perturbation Theory
• Addresses scenarios in which a minor parameter induces significant consequences, exemplified by boundary layers.
• Frequently employed in multi-scale systems requiring the differentiation of rapid and gradual dynamics.
• Beneficial in fields such as fluid dynamics, electrical circuits, and biological modelling.

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Theorem 4: Functional Analysis Methodology:
Functional analysis offers a mathematical framework for the examination of Volterra equations in infinite-dimensional environments.

Semigroup Theory:

• Expands the examination of differential equations to abstract function spaces such as Banach and Hilbert spaces.
Facilitates comprehension of the temporal progression of integro-differential systems.
• Utilized in diverse domains, encompassing viscoelasticity and thermal conduction issues.
Volterra Operator Theory examines the integral operator within Volterra equations as a functional operator.
• Analyses spectral properties to comprehend solution behavior and stability.
• Beneficial in formulating theoretical and numerical methods for resolving these equations.

Theorem 5: Numerical Approximation Methods:
Due to the infrequency of exact solutions, numerical approaches are essential for addressing Volterra integro-differential equations.

Collocation and Spectral Techniques:
• Highly accurate numerical methods utilizing polynomial approximations.
. Collocation methods impose solution conditions at discrete places, whereas spectral approaches employ global approximations.
• Exceptionally efficient for issues with continuous solutions.
Finite Difference and Finite Element Techniques
Transform equations into discrete formats for computational resolution.
Finite difference approaches employ grid-based approximations, whereas finite element methods utilize piecewise functions.
• Prevalently utilized in engineering and scientific simulations.

Theorem 6: Methods of Laplace Transform:
• Transforms integro-differential equations into algebraic equations inside the Laplace domain.
• Facilitates the resolution of linear equations utilizing established kernel functions.
• Necessitates an inverse Laplace transform to revert to the time domain.

Fractional-Order Volterra Equations:
These augment classical Volterra equations by integrating fractional derivatives,

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which more precisely include memory effects.
Caputo and Riemann-Liouville Derivatives delineate fractional-order derivatives in distinct manners, influencing the imposition of beginning conditions. These equations characterize systems exhibiting hereditary effects, including viscoelastic materials and anomalous diffusion. Solutions frequently incorporate specialized functions such as Mittag-Leffler functions, which extend the concept of the exponential function.

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